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Exact solution of generalized Tavis–Cummings models in quantum optics

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Abstract. Quantum inverse methods are developed for the exact solution of models which describe N two-level atoms interacting with one mode of the quantized electromagnetic field containing an arbitrary number of excitations M. Either a Kerr-type nonlinearity or a Stark-shift term can be included in the model, and it is shown that these two cases can be mapped from one to the other. The method of solution provides a general framework within which many related problems can similarly be solved. Explicit formulae are given for the Rabi splitting of the models for some N and M, on- and off-resonance. It is also shown that the solution of the pure Tavis–Cummings model can be reduced to solving a homogeneous ordinary differential equation of second order. Generalization of the method to the case of several cavity modes is indicated.

Much attention in quantum optics has been focussed on the Jaynes–Cummings (JC) model [1, 2] of one two-level atom coupled to a single mode of the quantized electromagnetic field in an ideal cavity. The JC model is an intrinsically nonlinear model. It is exactly solvable and, remarkably, has a physical realization in the one ^{85}Rb atom micromaser [3] and in the one ^{138}Ba atom microlaser [4]. The same model can also be realized [5] with spin polarized neutrons in a classical magnetic field.

In the case of the micromaser, the JC model is capable of correctly describing such features as Rabi oscillations, and such purely quantum phenomena as collapses and revivals of the atomic inversion, sub-Poissonian photon statistics and squeezing of the cavity field. If the cavity is not ideal because the cavity mode is coupled to a Kerr-like medium, an effective Hamiltonian can be derived [6] which adds a fourth-order term in the boson operators to the simple JC Hamiltonian. This model can also be solved exactly. Another effect which has been considered in this case of the micromaser is the Stark shift of the atomic levels, an effect caused by other (non-resonant) excitation levels of the atom [7].

It has been suggested (see the reference in [2]) that, in the inevitable presence of even a very small amount of black body radiation, the much sought after evolutions of the cavity field in the micromaser to a pure Fock state, a number state $|n\rangle$ of n photons in the field, would be enhanced by increasing the repetition rate of the input of atoms. However, this enhanced rate would substantially increase the probability of finding two or more atoms in the cavity at the same time [8]. This raises the question of the exact solution of the N two-level atom, one mode problem in an ideal cavity when $N > 1$. This model was solved at exact resonance by Tavis and Cummings [9], and at off-resonance by Hepp and Lieb [10]. Despite the recent [11] interest in this model, the solution of the general case

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with either the Kerr nonlinearity term or the Stark shift term included in the Hamiltonian is still lacking.

In this paper we report exact solution of the N two-level atom, one cavity-mode problem in an ideal cavity for an arbitrary number of excitations M in the system. It is evident that with this solution it is possible to describe experimental situations in much greater generality than has been achieved before. We can also include in our N-atom model either the Kerr nonlinearity or the Stark shift term. In fact, we shall show below that, within a constant, these two cases lead to identical results in each sector of the Hilbert space labelled by the number of excitations M and the 'Dicke cooperation number' $S \leqslant N/2$ [12] introduced below.

The solution reported here is found by applications of the quantum inverse method (QIM) [13, 14] developed first of all for integrable quantum field theories. These applications involve some extension of the conventional QIM and one purpose of this paper is to report this new analysis. One feature of it is the multiple number $K > 1$ of sets of solutions of the Bethe equations involved. Another is that the solution of the Tavis–Cummings (TC) model found by these methods will be shown to be equivalent to solving a homogeneous ordinary differential equation of second order. As for the quantum optics, we have successfully applied the QIM more conventionally to the *quantum* Maxwell–Bloch system [15], and there seems to be considerable scope for applying these various techniques to other models of significance in this field.

The most general Hamiltonian we consider in this paper is

$$\mathbf{H} = \mathbf{h}_0 + g(a^\dagger S^- + aS^+) + \gamma a^\dagger a^\dagger aa + \gamma (S^z)^2 \tag{1}$$

with $\mathbf{h}_0 \equiv \omega a^\dagger a + \omega_0 S^z$, and we solve it exactly up to determination of the roots of certain polynomials. In $\mathbf{H}$, g and γ are coupling constants, and $S^\pm = \sum_{n=1}^N \sigma_n^\pm$, $S^z = \frac{1}{2}\sum_{n=1}^N \sigma_n^z$ are collective N-atom Dicke operators (spin operators for which total spin $S \leqslant N/2$) satisfying the $su(2)$ Lie algebra $[S^+, S^-] = 2S^z, [S^z, S^\pm] = \pm S^\pm$. As usual $[a, a^\dagger] = 1$ for the single mode. When $\gamma = 0$, $\mathbf{H}$ reduces to the TC model and this becomes the JC model for $N = 1$ ($S = \frac{1}{2}$). When $S = \frac{1}{2}$, $(S^z)^2 = 1$ and $\mathbf{H}$ with $\gamma \neq 0$ is the JC model with Kerr nonlinearity.

The number operator

$$\mathbf{M} \equiv S^z + a^\dagger a \tag{2}$$

commutes with $\mathbf{H}$: $[\mathbf{H}, \mathbf{M}] = 0$. We can therefore set $\mathbf{H}_0 = g^{-1}(\mathbf{H} + (\gamma - \omega)\mathbf{M} - \gamma \mathbf{M}^2)$ and $[\mathbf{H}, \mathbf{H}_0] = 0$. We can then write

$$\mathbf{H}_0 = \Delta S^z + (a^\dagger S^- + aS^+) + ca^\dagger a S^z \tag{3}$$

where $c = -2g^{-1}\gamma$ and $\Delta = g^{-1}(\delta\omega + \gamma)$, with $\delta\omega = \omega_0 - \omega$, is the frequency shifted detuning. Note that the last term on the right-hand side of (3) causes photon number dependent changes in the atomic transitions and describes therefore a Stark shift. Henceforth we shall consider $\mathbf{H}_0$ and give its exact solution, but the same results can immediately be extended to the model with Kerr nonlinearity, Hamiltonian (1), through the mapping given above.

The total spin operator of the model is given by the Casimir operator $\mathbf{S}^2 = S^+S^- + S^z(S^z - 1)$: $[\mathbf{S}^2, \mathbf{H}_0] = 0$. Since $[\mathbf{M}, \mathbf{H}_0] = [\mathbf{S}^2, \mathbf{H}_0] = 0$, it is convenient (cf the corresponding analysis of the TC model in [9, 10]) to decompose the Hilbert space of the model, $\mathcal{H}_N = \mathcal{H}_B \otimes \mathbf{C}_1^2 \otimes \cdots \otimes \mathbf{C}_N^2$, in terms of the irreducible representations of $su(2)$ with

spin S and the excitation numbers M: $\mathcal{H}_N = \oplus_{S\geqslant 0}^{N/2} Y(N,S) \oplus_{M=0}^{\infty} \mathcal{H}(N,S,M-S)$. Here $Y(N,S) = N!(2S+1)\{(\frac{1}{2}N+S+1)!(\frac{1}{2}N-S)!\}^{-1}$ is the degeneracy of the representation $\mathcal{D}_S$ in the reduction of $(\mathcal{D}_{\frac{1}{2}})^N$ and S is the 'Dicke cooperation number' as mentioned above.

We thus have to solve the three simultaneous eigenvalue problems:

$$\begin{aligned} \mathbf{H}_0|\Phi^{\sigma}_{S,M}\rangle &= E^0_{S,M,\sigma}|\Phi^{\sigma}_{S,M}\rangle \\ \mathbf{S}^2|\Phi^{\sigma}_{S,M}\rangle &= S(S+1)|\Phi^{\sigma}_{S,M}\rangle \\ \mathbf{M}|\Phi^{\sigma}_{S,M}\rangle &= (M-S)|\Phi^{\sigma}_{S,M}\rangle \end{aligned} \tag{4}$$

where $0 \leqslant S \leqslant N/2$ and $0 \leqslant M < \infty$. Note that the quantum numbers M as defined in (4) are integers and in fact M is the number of excitations in the system; the σ are positive integers (see below). In the traditional approach [9] to the solution of the TC model it has been usual to look for the $|\Phi^{\sigma}_{S,M}\rangle$ in the form

$$|\Phi^{\sigma}_{S,M}\rangle = \sum_{m=0}^{J} A^{M}_{S,m,\sigma}(S^+)^m(a^\dagger)^{M-m}|\Omega_S\rangle \tag{5}$$

with $J = \min(2S, M)$. The vacuum state is $|\Omega_S\rangle = |0\rangle|S,-S\rangle$; $a|0\rangle = 0$; $S^-|S,-S\rangle = 0$, with $\mathbf{S}^2|S,-S\rangle = S(S+1)|S,-S\rangle$. The collective Dicke states generated from and including $|S,-S\rangle$ are the $|S,m\rangle$ for which $S^z|S,m\rangle = m|S,m\rangle, -S \leqslant m \leqslant$S. Note that in general there are several vacua for each $0 \leqslant S \leqslant N/2$ and the number of these is calculated in [12]. In equations (4), (5) the additional label $\sigma = 1, 2, \ldots, K$, $K = \min(2S, M) + 1$, is introduced [9], since the $|\Phi^{\sigma}_{S,M}\rangle$ are simultaneous eigenstates of the three commuting operators: then the $A^{M}_{S,m,\sigma}$ are coefficients to be determined. In effect equation (5) forms the 'dressed-state' basis [16] for fixed S, M; the σ label the energy splittings. In the dressed-state basis the rank of the Hamiltonian H_0 is K. We show that in the application of the QIM these labels σ define the K different sets of solutions of the Bethe equations (10) below.

In contrast to the traditional approach which is based on employing the ansatz (5), we shall use the QIM, which is an algebraic method such that the solution of the problem is intimately connected with the construction of a Hamiltonian. This method is often advantageous, because at the same time it provides the solution of a whole family of related problems. For reasons which will become apparent later we can apply what is already [13] a relatively new form of the QIM to the problem with the Hamiltonian $\mathbf{H}_0$ and number operator $\mathbf{M}$ by considering the two 2×2 matrix operators $\mathbf{L}_B(\lambda), \mathbf{L}_S(\lambda)$ for bosons and spins respectively [13]

$$\mathbf{L}_B(\lambda) = \begin{pmatrix} \lambda - \Delta - c^{-1} - ca^\dagger a & a^\dagger \\ a & -c^{-1} \end{pmatrix} \qquad \mathbf{L}_S(\lambda) = \begin{pmatrix} \lambda - cS^z & cS^+ \\ cS^- & \lambda + cS^z \end{pmatrix} \tag{6}$$

in which λ is a complex number. Note that the *elements* of $\mathbf{L}_B(\lambda)$ each commute with each of the elements of $\mathbf{L}_S(\lambda)$. For the QIM we need the *monodromy matrix* [13] $\mathbf{T}(\lambda)$ and for this object we can now set

$$\mathbf{T}(\lambda) = \mathbf{L}_B(\lambda)\mathbf{L}_S(\lambda) = \begin{pmatrix} \mathbf{A}(\lambda) & \mathbf{B}(\lambda) \\ \mathbf{C}(\lambda) & \mathbf{D}(\lambda) \end{pmatrix} \tag{7}$$

so that $\mathbf{A}(\lambda) = (\lambda - \Delta - c^{-1} - ca^\dagger a)(\lambda - cS^z) + cS^-a^\dagger$; $\mathbf{B}(\lambda) = \lambda\mathbf{X} - \mathbf{Y}$ with $\mathbf{X} = a^\dagger + cS^+$, $\mathbf{Y} = (1+c\Delta)S^+ - ca^\dagger S^z + c^2a^\dagger a S^+$ and $[\mathbf{X}, \mathbf{Y}] = 0$; $\mathbf{C}(\lambda) = -S^- + a(\lambda - cS^z)$; and $\mathbf{D}(\lambda) = caS^+ - c^{-1}(\lambda + cS^z)$.

The significant operator is the trace $\tau(\lambda) \equiv \mathrm{Tr}\,\mathbf{T}(\lambda)$ of the monodromy matrix $\mathbf{T}(\lambda)$: $\tau(\lambda) \equiv \mathbf{A}(\lambda) + \mathbf{D}(\lambda)$. In the QIM the operators $\tau(\lambda), \ln\tau(\lambda)$ are always generators of integrable (solvable) Hamiltonian operators. As can be checked $\tau(\lambda)$ determines $\mathbf{H}_0$ and $\mathbf{M}$ as $\mathbf{H}_0 = c^{-1}\tau(0)$; $\mathbf{M} = -c^{-1}(\partial\tau(\lambda)/\partial\lambda)_{\lambda=0} - c^{-1}\Delta - 2c^{-2}$, while the Casimir operator $\mathbf{S}^2$ is expressed through the quantum determinant [13] of $\mathbf{T}(\lambda)$ as $S^2 = D_q(0) = \mathrm{Det}_q\, T(0) = T(0)\sigma^2 T^{\mathrm{t}}(-c)\sigma^2$. Here σ^2 is a Pauli matrix and T^{t} means the transpose of T.

Note that, since $D_q(0)$ commutes with $\tau(\lambda)$, $[\mathbf{S}^2, \tau(\lambda)] = 0$. The operators $\tau(\lambda)$ satisfy $[\tau(\lambda), \tau(\mu)] = 0$ for *arbitrary* complex numbers λ, μ, and hence $\tau(\lambda)$ commutes with $\mathbf{H}_0$ and with $\mathbf{M}$, and the three (sets of) operators mutually commute. Then the eigenstates of $\tau(\lambda)$ are simultaneous eigenstates of $\mathbf{H}_0$ and $\mathbf{M}$ and in principle yield the eigenvalues of $\mathbf{H}_0$ and $\mathbf{M}$. The proof of the commutativity of the operators $\tau(\lambda)$ is based on the Yang–Baxter relation $\mathbf{R}(\lambda,\mu)\mathbf{T}(\lambda)\otimes\mathbf{T}(\mu) = \mathbf{T}(\mu)\otimes\mathbf{T}(\lambda)\mathbf{R}(\lambda,\mu)$, where $\mathbf{R}(\lambda,\mu) = c\mathbf{I} + (\lambda-\mu)\mathbf{P}$, and $\mathbf{P}$ is the permutation operator in $\mathbf{C}^2\otimes\mathbf{C}^2$ ($\mathbf{R}$ is the so-called rational R-matrix); $\mathbf{I}$ is a unit matrix.

It can be shown that $\mathbf{MB}(\lambda) = \mathbf{B}(\lambda)(\mathbf{M}+1)$, and likewise $\mathbf{S}^2\mathbf{B}(\lambda) = \mathbf{B}(\lambda)\mathbf{S}^2$. So as is already evident $\mathbf{B}(\lambda)$ acts as a *creation* operator for the quasi-particles of the theory while $\mathbf{C}(\lambda)$ is an annihilation operator. The M-particle states constructed in the usual fashion for the QIM method such that $|\Phi^\sigma_{S,M}(\lambda^\sigma_1, \lambda^\sigma_2, \ldots, \lambda^\sigma_M)\rangle = \prod_{j=1}^M \mathbf{B}(\lambda^\sigma_j)|\Omega_S\rangle = \prod_{j=1}^M(\lambda^\sigma_j\mathbf{X} - \mathbf{Y})|\Omega_S\rangle$, are eigenstates of $\tau(\lambda)$. These eigenstates are symmetric functions of their arguments λ^σ_j. Using $\mathbf{A}(\lambda)|\Omega_S\rangle = (\lambda - \Delta - c^{-1})(\lambda + cS)|\Omega_S\rangle, \mathbf{D}(\lambda)|\Omega_S\rangle = -c^{-1}(\lambda - cS)|\Omega_S\rangle$, it can further be shown that the λ^σ_j are in fact the roots of the so-called 'Bethe equations' which here take the form, for $n = 1, 2, \ldots, M$,

$$(1 + \Delta c - c\lambda^\sigma_n)\frac{\lambda^\sigma_n + cS}{\lambda^\sigma_n - cS} = \prod_{j\neq n}^{M}\frac{\lambda^\sigma_n - \lambda^\sigma_j + c}{\lambda^\sigma_n - \lambda^\sigma_j - c} \tag{8}$$

in which $0 \leqslant S \leqslant N/2$. Evidently there are K (modulo the permutation group) sets of solutions of these M Bethe equations the sets being labelled by σ, a new feature in the applications of the QIM as was already indicated. For $M = 1$ and $M = 2$ (at $\Delta = 0$) at least these equations can rather easily be solved analytically. For $M > 2$ we have solved them numerically for a few values of M, but these solutions will be reported elsewhere.

This way we can find the M-particle eigenenergies $E^0_{S,M,\sigma}$: $\mathbf{H}_0|\Phi^\sigma_{S,M}\rangle = E^0_{S,M,\sigma}|\Phi^\sigma_{S,M}\rangle$ and

$$E^0_{S,M,\sigma} = \frac{S}{c}\prod_{j=1}^{M}\left(1 - \frac{c}{\lambda^\sigma_j}\right) - \left(S\Delta + \frac{S}{c}\right)\prod_{j=1}^{M}\left(1 + \frac{c}{\lambda^\sigma_j}\right) \tag{9}$$

which together are the K eigenvalues of the Hamiltonian with Stark shift (3). From these $E^0_{S,M,\sigma}(\Delta, c)$ we can then find, through $c = -2g^{-1}\gamma$, $\Delta = g^{-1}(\delta\omega + \gamma)$, the K eigenvalues $E_{S,M,\sigma} = gE^0_{S,M,\sigma} - (\gamma - \omega)(M - S) + \gamma(M - S)^2$ of the Hamiltonian with Kerr nonlinearity (1). We have thus completely solved the eigenvalue problems of Hamiltonians (1) and (3) up to the solutions λ^σ_j of the Bethe equations (8) for each σ.

The TC model is obtained from (1) by $\gamma \to 0$. In this limit ($c \to 0$) from the terms linear in c the Bethe equations (8) become

$$\frac{2S}{\lambda^\sigma_n} - \lambda^\sigma_n + \Delta - \sum_{j=1}^{M}\frac{2}{\lambda^\sigma_n - \lambda^\sigma_j} = 0 \tag{10}$$

and we must continue to distinguish K sets of solutions: $\sigma = 1, 2, \ldots,$ K. The K M-photon eigenenergies given by the solutions λ_j^σ of (10) are $E^0_{S,M,\sigma}(\Delta, c = 0) = \Delta(M - S) - \sum_{j=1}^{M} \lambda_j^\sigma$, so that the $E_{S,M,\sigma}$ for the TC model are given by $E^{\mathrm{TC}}_{S,M,\sigma} = \omega(M - S) + gE^0_{S,M,\sigma}(\Delta = g^{-1}\delta\omega, c = 0)$. Note that the energy eigenvalues are given for each model for *arbitrary integer values* of the parameters $M \geqslant 0$ and $N > 0$.

As far as the physics goes as a particular consequence of these results the 'vacuum field Rabi splitting' [17], given by $M = 1$ photons for any number of atoms $N \geqslant 1$, is found from (9) in the form

$$\delta E_S = 2g\left[2S + (2g)^{-2}(\delta\omega + \gamma - 2\gamma S)^2\right]^{\frac{1}{2}} \tag{11a}$$

for the Kerr-nonlinearity Hamiltonian (1), and in the form

$$\delta E_S = 2\left[2S + \tfrac{1}{4}(\Delta + cS)^2\right]^{\frac{1}{2}} \tag{11b}$$

for the Stark-shift Hamiltonian (3). It is evident from equations (11*a*), (11*b*) that for finite detuning both the Stark shift and the Kerr nonlinearity give a noticeable effect only for rather more than one atom in the cavity. Here the quantum number $S \geqslant 0$ can have any value $S \leqslant N/2$ at unit intervals. For each number of atoms $N \geqslant 1$ there is therefore a series of Rabi shifts and splittings. The maximum Rabi splitting at exact resonance ($\omega_0 = \omega$) and for negligible Kerr nonlinearity ($\gamma = 0$) is therefore $2gN^{1/2}$, a result which is actually observed in [17]. In these experiments the initial state of the atoms in the cavity is such that only the $S = N/2$ states can be realized [18]. The energies of the $M > 1$ states can be calculated from (9) with (8) as easily as could the $M = 1$ states with splittings (11*a*), (11*b*). In particular, for $M = 2$ and $\gamma \to 0$, the Bethe equation (10) is very easily solved at exact resonance ($\delta\omega = 0$); for $2S = N > 1$ the energy levels $E^0_{S,2,\sigma}$ have $K = 3$ and form the triplet $2\omega_0; 2\omega_0 \pm 2g(N - \frac{1}{2})^{1/2}$, while for $M = 1$ the doublet has energies $\omega_0 \pm gN^{1/2}$. These results agree with those found by the traditional method for the TC model, namely that of [19] for $M = 2$ and that of [20] for $M = 1$.

We have shown [21] that the method of observation used in the experiments [17, 18], which uses the atoms as their own detectors via a weak RF probe, in effect measures the fluorescence spectrum. From the dipole matrix elements $\langle\Phi^\sigma_{S,M'}|S^+ + S^-|\Phi^\sigma_{S,M}\rangle$, the number M can only change by unity, so, for $N > 1$, there are six possible transitions for $M = 1 \to M = 2$ and six peaks at $\omega_0 \pm gN^{1/2}$ and $\omega_0 \mp gN^{1/2} \pm 2g(N - \frac{1}{2})^{1/2}$ superimposed on the doublet (for $M = 0 \to M = 1$) at $\omega_0 \pm gN^{1/2}$. Of these six peaks, four either lie on the doublet or inside it and, for finite peak widths, raise the observed minimum between the doublet peaks. The remaining two peaks form symmetrically placed sidebands at $\omega_0 \pm (gN^{1/2} + 2g(N - \frac{1}{2})^{1/2})$.

In the experiments [17] $N > 1$ and $N = 3$ and $N = 6$ in particular: For $N = 3$ we find the peak at $\omega_0 + g(\sqrt{3} + \sqrt{10})$ agrees in position with that observed in [17] for values of g corresponding to the observed doublet separation. For $N = 3$ and $M \geqslant 2$, $K = 3$ and only triplets occur; e.g., for $M = 2 \to M = 3$ transitions there are nine possible transitions including one at ω_0 within the doublet. Such transitions are not observable as such within the present experimental accuracy at the finite temperatures.

For the case of one atom ($S = \frac{1}{2}$) and general M, $K = 2$ and we find the spectrum which for $\delta\omega = 0$ coincides with that given in the [21, figure 12], while the emergence of these peaks in the observable microwave spectrum at finite temperatures is already studied numerically by matrix continued fraction methods in [21–24]. The result for the data which

is essentially that of the experiments [17] is given in [22, figure 3], which also shows sidebands. Note again how the K sets of solutions of the Bethe equations also determine the g dependent level shifts and splittings. These are the origin of the various sidebands in the observable spectrum for $N = 1$ and $N > 1$—a remarkable connection between BA theory and experiment.

It is perhaps instructive at this point to return to the mathematics and establish contact between the quantum inverse solution and that of the traditional ansatz (5). Following [13] we can express [25] the amplitudes $A^M_{S,m,\sigma}$ in the ansatz through the solutions of the Bethe equations (8):

$$A^M_{S,m,\sigma} = \sum \prod_{j\in \mathrm{I}} \prod_{k\in \mathrm{II}} (c\lambda^\sigma_j - \Delta c - 1)(\lambda^\sigma_k - cS)\left(1 - \frac{c}{\lambda^\sigma_j - \lambda^\sigma_k}\right).$$

Here summation is over all decompositions of the solutions of the Bethe equations, $\{\lambda^\sigma\} = \{\lambda^\sigma_{\mathrm{I}}\} \cup \{\lambda^\sigma_{\mathrm{II}}\}$, such that the intersection of the two subsets is zero and there are $M - m$ elements in subset I and m elements in subset II.

There is also another way to find the solutions of the Bethe equations and the amplitudes $A^M_{S,m,\sigma}$. We shall describe this method for the pure TC model for which $\gamma = 0$ ($c = 0$). In this case $\mathbf{B}(\lambda) = \lambda a^\dagger - S^+$, and by expressing the $|\Phi^\sigma_{S,M}\rangle$ as $|\Phi^\sigma_{S,M}\rangle = \prod^M_{j=1}(\lambda^\sigma_j a^\dagger - S^+)|\Omega_S\rangle = \sum_m A^M_{S,m,\sigma}(a^\dagger)^{M-m}(S^+)^m|\Omega_S\rangle$, which makes the connection with the traditional ansatz (5) particularly transparent, we can easily conclude that the amplitudes $A^M_{S,m,\sigma}$ are in fact the coefficients of the polynomial $\mathcal{P}(\lambda) = \prod^M_{j=1}(\lambda^\sigma_j - \lambda)$, i.e. $\mathcal{P}(\lambda)$ is the generating function of amplitudes $A^M_{S,m,\sigma}$. By using the Bethe equations (10), we can then prove that this polynomial satisfies an ordinary differential equation (ODE) of second order

$$\lambda\mathcal{P}'' + (\lambda^2 - \Delta\lambda - 2S)\mathcal{P}' + (\zeta - M\lambda)\mathcal{P} = 0 \tag{12}$$

where the eigenvalues $\zeta = \zeta^\sigma_{S,M}, \sigma = 1, \ldots, K$, are determined by the condition that (12) has only polynomial solutions without multiple zeros: $\mathcal{P}(\lambda) = \mathcal{P}^\sigma_{S,M}(\lambda)$ and M is the degree of the polynomial. The eigenenergies of the TC model are given by $E^{\mathrm{TC}}_{S,M,\sigma} = g\zeta^\sigma_{S,M}(\Delta = g^{-1}\delta\omega) - S\delta\omega + \omega(M - S)$. Note that the zeros of the polynomial $\mathcal{P}(\lambda)$ as determined by the ODE (12) are the solutions of the Bethe equations (10). We can conclude that in this case of the TC model the Bethe ansatz solution of the model is equivalent to solving a homogeneous ODE. In the more general case with finite c (or γ) the ODE (12) is replaced by a more complicated functional equation.

As an example of the use of the generating function $\mathcal{P}(\lambda)$ we shall show how the spectrum of the JC model ($S = \frac{1}{2}$), $E^{\mathrm{JC}}_{M,\sigma} = E^{\mathrm{TC}}_{1/2,M,\sigma}$, can be derived from it. Differentiating (12) and noting that $E^0_{1/2,M,\sigma}(\Delta, c = 0) + \Delta/2 = \mathcal{P}'(0)/\mathcal{P}(0)$, we immediately find that $E^{\mathrm{JC}}_{M,\sigma} = \omega(M - \frac{1}{2}) \pm ((\delta\omega/2)^2 + g^2M)^{1/2}$. The same result can of course be derived from the Bethe equations (10) directly.

The solutions of the eigenvalue problems as described above determine the time evolutions in the usual way. Thus the evolution of the collective inversion $\langle|S^z(t)|\rangle$, the photon number $\langle|a^\dagger a|\rangle$ and the quantity $\langle|a^\dagger a^\dagger aa|\rangle - (\langle|a^\dagger a|\rangle)^2$ can all be expressed in terms of the amplitudes $A^M_{S,m,\sigma}$, and all of the observable dynamics calculated. The actual forms of the expressions are cumbersome, however, and these will be reported elsewhere. Evidently evolution of the actual photon statistics of the cavity field can also be found completely.

We have thus been able to use the quantum inverse method to solve for all aspects of the model system which has the Hamiltonian (1), or equivalently the Hamiltonian (3), and which

has the TC N-atom, one-mode model lying within it. For the solutions of the TC model the problem is also reduced to finding the zeros of the polynomial $\mathcal{P}(\lambda)$ which satisfies a homogeneous ordinary differential equation (12). The efficiency of the method applied here is also manifested by the fact that many other solvable models, in addition to those considered here, can be constructed by studying the representation theory of the **L** operators which satisfy the Yang–Baxter relation. Boson representations of the spin operators can, for example, be used to produce solvable three boson Hamiltonians. One example of these is solved in [26]. We have also generalized the Hamiltonians (1) and (3) to the case of several cavity modes, and to the case of multiphoton transitions. At present we have a formulation for these models at exact resonance, and we hope to report further details in this connection, and likewise for the models (1) and (3), elsewhere.

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